

# Minimum Enclosing Area Triangle with a Fixed Angle

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## Abstract

Given a set  $S$  of  $n$  points in the plane and a fixed angle  $0 < \omega < \pi$ , we show how to find all triangles of minimum area with angle  $\omega$  that enclose  $S$  in  $O(n \log n)$  time. We also demonstrate that in general, the solution cannot be written without cubic root.

## 1 Introduction

In geometric optimization, the goal is often to find an optimal object or optimal placement of an object subject to a number of geometric constraints (see [AS98] for a survey). Examples include finding the smallest circle enclosing a point set [Meg83, Wel91] or finding the smallest circle enclosing at least  $k$  points of a point set of  $n$  points ( $k \leq n$ ) [ESZ94, Mat95]. In our setting, we study the following problem: given a set  $S$  of  $n$  points in the plane, find all the triangles of minimum area with a fixed angle  $\omega$ ,  $0 < \omega < \pi$ , that enclose  $S$ . When no constraint is put on the angles, Klee and Laskowski [KL85] gave an  $O(n \log^2 n)$  time algorithm for finding the minimum area enclosing triangle. This was later improved to  $O(n \log n)$  by O'Rourke et al. [OAMB86] which is optimal. Bose et al. [BMSSed] provided optimal algorithms for the setting where one wishes to find the minimum area isosceles triangles. The setting we explore here is in between the two. We place a restriction on the angle but do not insist on the triangle to be isosceles. Our solution, which we outline below, uses ideas from the solutions of Klee and Laskowski and Bose et al.

The five main steps of the algorithm are presented in Sections 2, 3, 4, 5 and 6. Each section outlines one step, proves the mathematical formulas involved and gives its time complexity. As we present in Section 5, at some point, the algorithm needs to calculate the roots of a fourth degree polynomial. This is unfortunate when it comes to numerical robustness. However, in Section 7, we show that we cannot dodge such algebraic expressions. We finally conclude and present some related open problems in Section 8.

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## 2 Overview and Preliminaries

Since the solution to the general problem only needs to consider the vertices of the convex hull of  $S$ , the first step is to compute the convex hull. In the remainder of the paper, we assume that the input is a convex  $n$ -gon with vertices given in clockwise order.

Let  $P$  be a convex  $n$ -gon (refer to Figure 1). We denote the edges and the vertices of  $P$

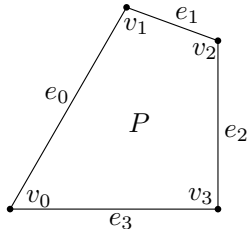


Figure 1: In this example,  $P$  is a quadrilateral and  $\omega = \frac{1}{2}\pi$ .

in clockwise order by  $e_i$  and  $v_i$  for  $0 \leq i \leq n-1$  (all index manipulation is modulo  $n$ ). Here and in the following sections, as we present the algorithm, we trace each step through the example of Figure 1.

We begin with two definitions.

**Definition 1** ( $\omega$ -wedge). *Let  $q$  be a point in the plane and  $\omega$  be an angle ( $0 < \omega < \pi$ ). Let  $\Delta$  and  $\Delta'$  be two rays emanating from  $q$  such that the angle between  $\Delta$  and  $\Delta'$  is  $\omega$ . We say that the closed set formed by  $q$ ,  $\Delta$ ,  $\Delta'$  and the points between  $\Delta$  and  $\Delta'$  creates an  $\omega$ -wedge, denoted  $\mathcal{W}(\omega, q, \Delta, \Delta')$ . The point  $q$  is called the apex of the  $\omega$ -wedge. An  $\omega$ -wedge  $W$  encloses a polygon  $P$  when  $P \subseteq W$  and both  $\Delta$  and  $\Delta'$  are tangent to  $P$ .*

**Definition 2** ( $\omega$ -cloud). *Let  $P$  be a convex  $n$ -gon and  $\omega$  be an angle ( $0 < \omega < \pi$ ). By rotating an enclosing  $\omega$ -wedge around  $P$ , the apex traces a sequence of circular arcs that we call an  $\omega$ -cloud (refer to Figure 2).*

There are many technical details involved in the solution to this problem. Before getting too caught up in the technical details, let us first review the general approach to our solution.

Since we only consider enclosing triangles with an angle of  $\omega$ , each optimal triangle can be constructed from an  $\omega$ -wedge that encloses  $P$ . Therefore, we consider all possible  $\omega$ -wedges that enclose  $P$ . The apices of these  $\omega$ -wedges lie on an  $\omega$ -cloud  $\Omega$  which consists of a linear number of pieces of circular arcs (refer to [BMSSed]). Then, for each of these  $\omega$ -wedges, we find the minimal triangle by identifying a third side. For the triangle to be optimal, the midpoint of this third side has to touch  $P$  (see Proposition 3 and Corollary 6). Hence, for each enclosing  $\omega$ -wedge, there is one and only one triangle to consider for optimality.

Moreover, when the apex  $q$  of an  $\omega$ -wedge  $W$  moves clockwise around the  $\omega$ -cloud  $\Omega$ , the midpoint  $m$  of the third side of the optimal triangle moves clockwise around  $P$  (see Lemma 7). As  $W$  moves, we note the positions of  $q$  where  $m$  leaves an edge of  $P$  and where  $m$  enters a new edge of  $P$ . These positions of  $q$  are important event points. They divide  $\Omega$  into a linear number of sections (see Section 4). Let  $S$  be one of these sections. We prove

that the minimum triangle having a vertex (hence an angle  $\omega$ ) on  $S$  can be computed in constant time (see Lemmas 10, 11, 12 and 13). We then have a linear number of candidates (one for each section of  $\Omega$ ) to consider rather than infinitely many if we had to take all possible enclosing  $\omega$ -wedges into account. What remains is to identify the optimal ones from these linear candidates. With this in mind, the technical details will fall into place.

**Step 1** Compute the  $\omega$ -cloud around  $P$  and denote it by  $\Omega$ .

$\Omega$  consists of  $n' = O(n)$  circular arcs that we denote in clockwise order by  $\Gamma_i$  for  $0 \leq i \leq n' - 1$ . The intersection point of  $\Gamma_i$  and  $\Gamma_{i+1}$  is denoted by  $u_{i+1}$  for  $0 \leq i \leq n' - 1$  (refer to Figure 2). We also refer to  $\Gamma_i$  by the closed set  $[u_i, u_{i+1}]$  containing all the points of  $\Gamma_i$ . **Step 1** takes

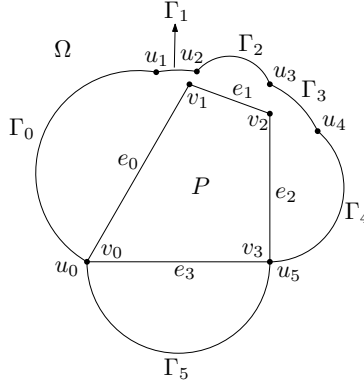


Figure 2: **Step 1**:  $\Omega$  is the  $\frac{1}{2}\pi$ -cloud of  $P$  ( $n' = 6$ ).

$O(n)$  (refer to [BMSSed]).

### 3 Optimal Solution Given a Fixed $\omega$ -Wedge

In this section, we present a routine that computes the minimum enclosing area triangle from a fixed  $\omega$ -wedge enclosing  $P$  (see Algorithm 4). Then we describe **Step 2** that uses this routine.

Take any  $\omega$ -wedge  $W = \mathcal{W}(\omega, q, \Delta, \Delta')$  that encloses  $P$ . Find  $b \in \Delta$  (respectively  $c \in \Delta'$ ) such that  $P$  is enclosed in  $\triangle qbc$  and the midpoint  $m$  of  $bc$  is on  $P$ . We claim that

1. it is always possible to find  $b$  and  $c$  satisfying these properties,
2.  $\triangle qbc$  is the minimum triangle enclosing  $P$  that can be constructed with  $\mathcal{W}(\omega, q, \Delta, \Delta')$ ,
3. it takes  $O(n)$  time to compute  $\triangle qbc$ .

These claims are proven in Proposition 3, Algorithm 4, Lemma 5 and Corollary 6.

Proposition 3 addresses the case where  $q$  is outside of  $P$ . It was first proven by Klee and Laskowski (refer to [KL85]). We propose a slightly different proof. Among other things, we stress the fact that  $\triangle qbc$  always exists.

**Proposition 3.** *Let  $P$  be a convex  $n$ -gon and  $q$  be a point outside of  $P$ . Let  $\triangle qbc$  be the triangle that circumscribes  $P$  such that the midpoint  $m$  of segment  $bc$  lies on  $P$ . The following is true:*

1.  $\triangle qbc$  exists and is unique (hence it is well-defined).
2. Among all triangles that circumscribe  $P$  and have  $q$  as a vertex,  $\triangle qbc$  is the unique one of minimum area.

PROOF:

1. Take  $b$  and  $c$  on the rays that support  $P$  from  $q$  such that an edge  $e$  of  $P$  lies on  $bc$  (refer to Figure 3). If  $m \in e$ , then we are done. If not, suppose without loss of generality

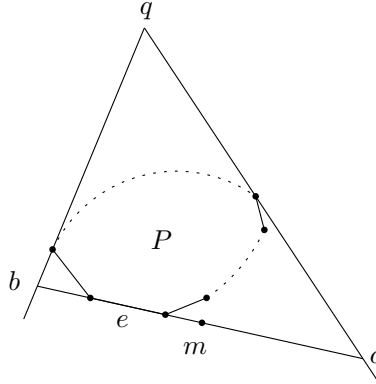


Figure 3:  $\triangle qbc$  exists.

that  $m$  is between  $e$  and  $c$ . Move  $b$  along its ray such that  $b$  gets farther from  $q$  and  $bc$  stays tangent to  $P$ . Therefore  $c$  gets closer to  $q$ . Since  $m$  moves continuously as  $b$  moves along its ray,  $m$  will eventually touch one of the  $P$ 's edges. This continuity argument implies the existence of  $\triangle qbc$ .

Suppose there are two triangles, namely  $\triangle qbc$  and  $\triangle qb'c'$  with midpoints  $m$  and  $m'$  respectively. We show that these two triangles are equal. There are three cases to consider: (1)  $m = m'$ , (2)  $m \neq m'$  and both are on the same edge of  $P$  or (3)  $m \neq m'$  and both are on different edges of  $P$ .

- (1) If  $m = m'$  is not a vertex of one of the edges of  $P$ , then it lies on the interior of one of the edges of  $P$  (refer to Figure 4). Hence this aforementioned edge is in  $bc \cap b'c'$ . This implies that  $b = b'$  and  $c = c'$  by construction.

Suppose  $m = m'$  is the vertex of one of the edges of  $P$ . Triangles  $\triangle mbb'$  and  $\triangle mcc'$  are congruent by the following: (refer to Figure 5).

- $|b'm| = |c'm|$  since  $m$  is the midpoint.
- $\angle b'mb = \angle c'mc$  since they are vertical angles.
- $|bm| = |cm|$  since  $m$  is the midpoint.

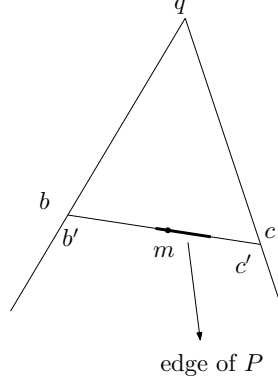


Figure 4:  $m = m'$  is not a vertex of  $P$ .

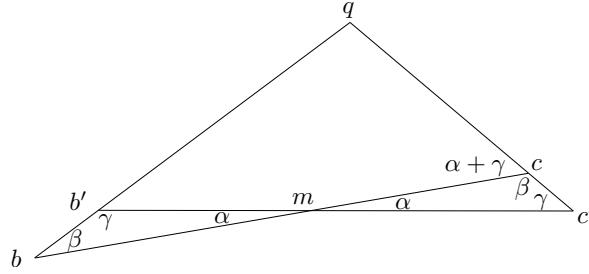


Figure 5:  $m = m'$  is a vertex of  $P$ .

Therefore  $\angle mbb' = \angle mcc'$  and  $\angle mb'b = \angle mc'c$ . Hence we have the following:

$$\begin{aligned}\alpha &= \angle b'mb = \angle c'mc, \\ \beta &= \angle mbb' = \angle mcc', \\ \gamma &= \angle mb'b = \angle mc'c,\end{aligned}$$

where  $\alpha + \beta + \gamma = \pi$ . So  $\angle bcq = \alpha + \gamma$  because  $\angle bcq$  and  $\angle c'cb$  are supplementaries. Therefore  $\angle bqc = \pi - \alpha - \beta - \gamma = 0$ , which is impossible unless  $\alpha = 0$ . We conclude that  $b = b'$  and  $c = c'$ .

- (2) Suppose  $m \neq m'$  and they are on the same edge of  $P$ . If neither  $m$  nor  $m'$  is a vertex, then this situation is similar to the one in Figure 4, hence  $b = b'$ ,  $c = c'$  and  $m = m'$ . This is a contradiction so this situation is impossible.

Without loss of generality, suppose that  $m'$  is a vertex of  $P$  and that  $|m'b| < |m'c|$ . Let  $b''$  be the point on the line through  $b'c'$  such that Segments  $bb''$  and  $cc'$  are parallel. (refer to Figure 6). Triangles  $\triangle m'bb''$  and  $\triangle m'cc'$  are similar by the following:

- $\angle bm'b'' = \angle cm'c'$  since they are vertical angles.
- $\angle m'bb'' = \angle m'cc'$  since they are alternate angles.

Therefore, since  $|m'b| < |m'c|$ , then  $|m'b''| < |m'c'|$ . However, since  $m'$  is  $b'c'$ 's midpoint,  $|m'b''| > |m'c'|$ . This is a contradiction so this situation is impossible.

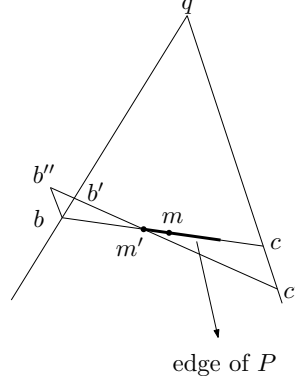


Figure 6:  $m'$  is a vertex of  $P$ .

- (3) Suppose  $m \neq m'$  are not on the same edge of  $P$ . This case is similar to the case where  $m \neq m'$  and both points are on the same edge of  $P$ .
2. Let  $\triangle qb'c'$  be a triangle of minimum area that circumscribes  $P$  and has  $q$  as a vertex (refer to Figure 7). Suppose for a contradiction that the midpoint  $m'$  of  $b'c'$  does not

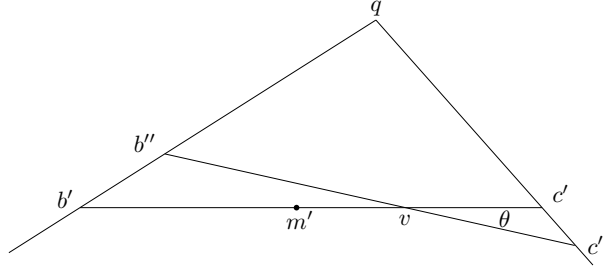


Figure 7:  $\triangle qbc$  has minimum area.

lie on one of the edges of  $P$  (refer to Figure 7).

Since  $P$  is convex and  $\triangle qb'c'$  circumscribes  $P$ , the following construction is possible. Let  $v \in b'c'$  be the vertex of  $P$  closest to  $m'$ . Without loss of generality, suppose  $vc' < vb'$ . Let  $b''c''$  be a line segment that goes through  $v$  and such that  $b'' \in qb'$  and  $c''$  lies on the line through  $q$  and  $c'$ . Let  $\theta = \angle c'vc''$ .  $\angle b'vb'' = \angle c'vc''$  since they are vertical angles. If  $\theta$  is sufficiently small, then  $vc'' < vb''$  and  $\triangle qb''c''$  circumscribes  $P$ .

Therefore

$$\begin{aligned}
 \text{area}(\triangle qb'c') &= \text{area}(\triangle b'vb'') + \text{area}([qb''vc']) \\
 &= \frac{1}{2}|vb'| |vb''| \sin(\theta) + \text{area}([qb''vc']) \\
 &> \frac{1}{2}|vc'| |vc''| \sin(\theta) + \text{area}([qb''vc']) \\
 &= \text{area}(\triangle c'vc'') + \text{area}([qb''vc']) \\
 &= \text{area}(\triangle qb''c'')
 \end{aligned}$$

which contradicts the fact that  $\triangle qb'c'$  has minimum area. So  $m'$  lies on one of the edges of  $P$  and  $\triangle qb'c'$  is a local minimum among triangles circumscribing  $P$  and having  $q$  as a vertex. We proved in Point 1 that there exists one and only one such triangle.  $\square$

Given the setting of Proposition 3, we show how to compute  $\triangle qbc$  in  $O(n)$  time. In Algorithm 4, the edges of  $P$  are considered in clockwise order. The variable *edge* indicates the edge of  $P$  that is currently being considered. Algorithm 4 uses Lemma 5 that is presented after this routine.

**Algorithm 4.** (*Minimum Enclosing Triangle with a Fixed  $\omega$ -Wedge*)

- *INPUT: An  $n$ -gon  $P$  and an enclosing  $\omega$ -wedge  $W = \mathcal{W}(\omega, q, \Delta, \Delta')$  that supports  $P$  at vertices  $v_0$  and  $v_k$  (refer to Figure 8).*

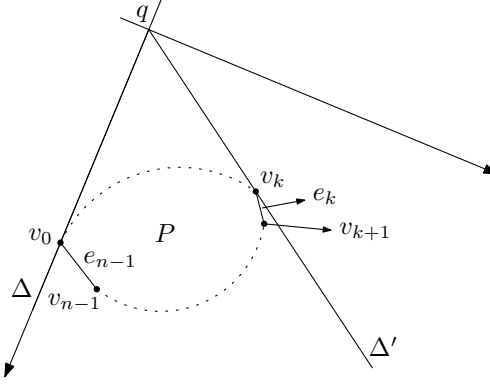


Figure 8: Computing  $\triangle qbc$  takes  $O(n)$  time.

- *OUTPUT: The minimum triangle enclosing  $P$  that can be constructed with  $\mathcal{W}(\omega, q, \Delta, \Delta')$ .*
1.  $edge \leftarrow e_k$ .
  2. *If the line through edge does not intersect  $\Delta$  or  $\Delta'$ ,*
    - $edge \leftarrow next(edge)$ .
    - *Go to 2.*
  3. *Let  $b$  (respectively  $c$ ) be the intersection point of the line through edge and  $\Delta$  (respectively  $\Delta'$ ). If the midpoint of  $bc$  is on  $P$ , return  $\triangle qbc$ .*
  4. *If the midpoint of  $bc$  is to the left of edge,*
    - $edge \leftarrow next(edge)$ .
    - *Go to 3.*

5. If the computation reaches this step, it means that the midpoint of  $bc$  is to the right of edge.

- Place the cartesian coordinate system on  $W$  such that  $q = (0,0)$  and  $\Delta$  is the positive  $x$ -axis (refer to Figure 8).
- Let  $v = (s, t) := \text{edge} \cap \text{previous}(\text{edge})$ .
- Take  $b = (2(s - t \cot(\omega)), 0)$  and  $c = (2t \cot(\omega), 2t)$  (refer to the proof of Lemma 5).
- Return  $\Delta qbc$ .

In the worst case, Algorithm 4 considers all the edges of  $P$ . Since it spends  $O(1)$  time per edge, it takes  $O(n)$  time total.

**Lemma 5.** Let  $\Delta$  and  $\Delta'$  be two lines intersecting at  $q$  and let  $\omega$  ( $0 < \omega < \pi$ ) be the angle between  $\Delta$  and  $\Delta'$ . Let  $v$  be a point within  $\mathcal{W}(\omega, q, \Delta, \Delta')$  ( $v \notin \Delta$  and  $v \notin \Delta'$ ). In  $O(1)$  time, we can compute  $b \in \Delta$  and  $c \in \Delta'$  such that  $b, v$  and  $c$  lie on a single line and  $v$  is the midpoint of  $bc$ .

PROOF: Without loss of generality, suppose  $\Delta$  is the  $x$ -axis,  $\Delta'$  is the line<sup>1</sup>  $x - \cot(\omega)y = 0$  and let  $v = (s, t)$  ( $t > 0$  and  $s - \cot(\omega)t \neq 0$ ) (refer to Figure 9).

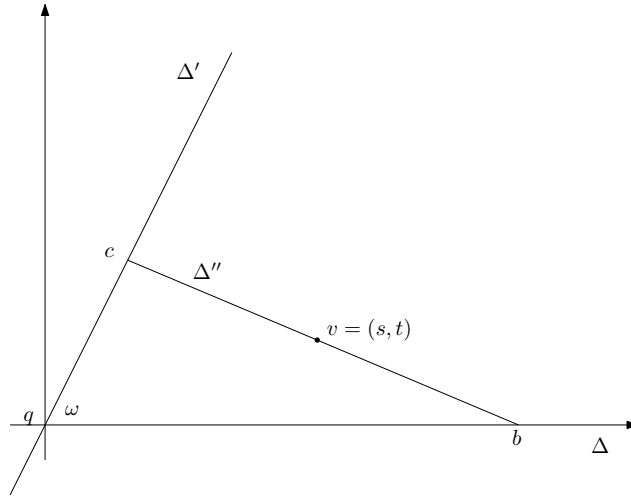


Figure 9: Proof of Lemma 5.

Let  $\Delta''$  be a line containing  $v$  such that  $\Delta$  is not parallel to  $\Delta''$  and  $\Delta'$  is not parallel to  $\Delta''$ . The line  $\Delta''$  is the one containing the points  $b$  and  $c$  we are looking for. There are two cases to consider: (1)  $\Delta''$  is not vertical or (2)  $\Delta''$  is vertical.

- (1) Suppose  $\Delta''$  is not vertical. Let  $b$  be the intersection point of  $\Delta$  and  $\Delta''$ , and  $c$  the intersection point of  $\Delta'$  and  $\Delta''$ . The general equation of  $\Delta''$  is  $y = \lambda(x - s) + t$

<sup>1</sup> In what follows, several algebraic expressions are written using  $\cot(\cdot)$  where it seems that they could be simplified by using  $\tan(\cdot)$  instead. However, we write  $\cot(\cdot)$  in order to properly deal with the angle  $\frac{1}{2}\pi$ .



for  $\lambda \neq 0$  and  $\frac{1}{\lambda} \neq \cot(\omega)$ . Calculating the intersection point of  $\Delta$  and  $\Delta''$ , and of  $\Delta'$  and  $\Delta''$ , we find that the general coordinates of  $b$  and  $c$  are  $b = (\frac{\lambda s - t}{\lambda}, 0)$  and  $c = (\frac{t - \lambda s}{1 - \lambda \cot(\omega)} \cot(\omega), \frac{t - \lambda s}{1 - \lambda \cot(\omega)})$ . We are looking for  $\lambda$  such that  $|bv| = |vc|$ , so we need to isolate  $\lambda$  in

$$\left(\frac{\lambda s - t}{\lambda} - s\right)^2 + (0 - t)^2 = \left(\frac{t - \lambda s}{1 - \lambda \cot(\omega)} \cot(\omega) - s\right)^2 + \left(\frac{t - \lambda s}{1 - \lambda \cot(\omega)} - t\right)^2$$

which solves to  $\lambda = \frac{t}{2t \cot(\omega) - s}$ . Therefore  $b = (2(s - t \cot(\omega)), 0)$  and  $c = (2t \cot(\omega), 2t)$ .

Note that  $\lambda \neq 0$  because  $t \neq 0$  and  $\frac{1}{\lambda} \neq \cot(\omega)$  because  $s - \cot(\omega)t \neq 0$ .

- (2) Suppose  $\Delta''$  is vertical. Therefore  $0 < \omega < \frac{1}{2}\pi$ , otherwise  $\Delta''$  is not the expected line. Using the notation of the previous case, we get that the equation of  $\Delta''$  is  $x = s$ . Moreover,  $b = (s, 0)$  and  $c = (s, s \tan(\omega))$ . We are looking for  $|bv| = |vc|$ , which means

$$(s - s)^2 + (0 - t)^2 = (s - s)^2 + (s \tan(\omega) - t)^2.$$

So this situation occurs if and only if  $2t = s \tan(\omega)$  and the solution is  $b = (s, 0) = (2(s - t \cot(\omega)), 0)$  and  $c = (s, s \tan(\omega)) = (2t \cot(\omega), 2t)$ .

Hence the global solution is  $b = (2(s - t \cot(\omega)), 0)$  and  $c = (2t \cot(\omega), 2t)$  in all cases.  $\square$

Note that what we obtained is more general than our claims. Because  $q$  is outside of  $P$ , Proposition 3 together with Algorithm 4 and Lemma 5 not only compute the minimum triangle enclosing  $P$  that can be constructed with  $\mathcal{W}(\omega, q, \Delta, \Delta')$ , it also computes the minimum triangle enclosing  $P$  and having  $q$  as a vertex.

The case where  $q$  is on  $P$  is similar. We conclude with it.

**Corollary 6.** *Consider an  $\omega$ -wedge  $W = \mathcal{W}(\omega, q, \Delta, \Delta')$  enclosing  $P$  such that  $q$  is on  $P$ . The minimum triangle enclosing  $P$  that can be constructed with  $W$  can be computed in  $O(n)$  time.*

Therefore, given any fixed  $\omega$ -wedge  $W$  enclosing  $P$ , we can compute the minimum enclosing area triangle that can be constructed with  $W$ . The midpoint  $m$  of the third side of this optimal triangle has to be on  $P$ . In **Step 3** (see Section 4), we need to know the exact position of  $m$  for one fixed  $\omega$ -wedge enclosing  $P$ . However, it does not matter for what enclosing  $\omega$ -wedge we know the position of  $m$ , as soon as we know for one. Therefore, in **Step 2**, we fix an arbitrary  $\omega$ -wedge  $W$  enclosing  $P$  and we compute the position of  $m$ .

**Step 2** Let  $q \in \Gamma_0$  be such that  $q = u_0$ . Consider the  $\omega$ -wedge  $W = \mathcal{W}(\omega, q, \Delta, \Delta')$  that encloses  $P$  (refer to Figure 10). Apply Algorithm 4 with  $P$  and  $W$ .

Consider Figure 2. Any  $\omega$ -wedge enclosing  $P$  and having its vertex in  $\Gamma_0 \setminus \{u_0, u_1\}$  is such that  $v_0 \in \Delta'$  and  $v_1 \in \Delta$ . This property easily generalizes to the case where the apex of the  $\omega$ -wedge is on  $u_0$  or on  $u_1$ . Therefore, when we specify  $q \in \Gamma_0$  and  $q = u_0$ , we imply



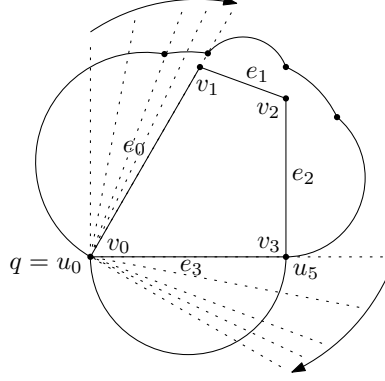


Figure 11:  $\frac{1}{2}\pi$ -wedge turning clockwise around a corner.

is possible to move  $q$  such that  $m$  stays on  $e$ . Such a displacement of  $q$  has to be of less than  $\varepsilon$  for an  $\varepsilon > 0$  and such an  $\varepsilon$  always exists.

What needs to be shown is that  $m$  moves clockwise around  $\Omega$ .

**Lemma 7.** *As an  $\omega$ -wedge  $W = \mathcal{W}(\omega, q, \Delta, \Delta')$  moves clockwise such that its apex  $q$  stays on  $\Omega$ , the midpoint  $m$  of  $bc$  —the third edge of the minimum triangle enclosing  $P$  that can be constructed with  $W$ — moves clockwise along  $P$ . Specifically, take  $q', q'' \in \Omega$  with  $q''$  clockwise from  $q'$ .*

1. *If  $q' \in \Gamma_i \setminus \{u_{i+1}\}$  ( $0 \leq i \leq n' - 1$ ),  $q'' \in ]q', u_{i+1}]$ , and  $q'$  and  $q''$  are close enough to each other, then we have the following:*

- (a) *If  $m' \in e_j \setminus \{v_j, v_{j+1}\}$  ( $0 \leq j \leq n - 1$ ), then  $m'' \in ]m', v_{j+1}] \subset e_j$ .*
- (b) *If  $m' = v_j$  ( $0 \leq j \leq n - 1$ ), then  $m'' \in e_j$  (possibly  $m'' = m'$ ).*

2. *If  $q' = q'' = u_i$  ( $0 \leq i \leq n' - 1$ ) and  $u_i = v_l$  ( $0 \leq l \leq n - 1$ ). Let  $W'$  and  $W''$  be two different  $\omega$ -wedges enclosing  $P$  with  $q' = q'' = u_i$  as an apex. If  $W''$  is clockwise from  $W'$  (refer to Figure 11) and  $W'$  and  $W''$  are close enough to each other, then we have the following:*

- (a) *If  $m' \in e_j \setminus \{v_j, v_{j+1}\}$  ( $0 \leq j \leq n - 1$ ), then  $m'' \in ]m', v_{j+1}] \subset e_j$ .*
- (b) *If  $m' = v_j$  ( $0 \leq j \leq n - 1$ ), then  $m'' \in e_j$  (possibly  $m'' = m'$ ).*

PROOF: Without loss of generality,  $b'c'$  is on the  $x$ -axis (refer to Figure 12).

1. (a) If  $q'$  and  $q''$  are close enough to each other, then  $m'' \in e_j$ . Therefore  $b'' = (b''_x, 0)$ ,  $b' = (b'_x, 0)$ ,  $c'' = (c''_x, 0)$  and  $c' = (c'_x, 0)$  with  $b''_x < b'_x < c''_x < c'_x$ . Hence  $m'' = \frac{b''_x + c''_x}{2} < \frac{b'_x + c'_x}{2} = m'$ , so  $m'' \in ]m', v_{j+1}] \subset e_j$ .
- (b) If  $q'$  and  $q''$  are close enough to each other, then the only situation to discard is  $m'' \in e_{j-1} \setminus \{v_j\}$ . So suppose  $m'' \in e_{j-1} \setminus \{v_j\}$  for a contradiction. Therefore  $m'$  and  $m''$  both belong to  $e_{j-1}$ . Hence an argument similar to the one of the previous case leads to  $m'' = v_j$ , which is a contradiction.

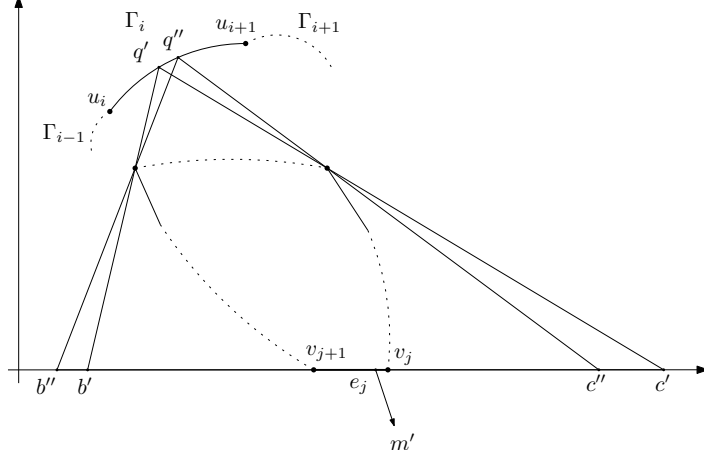


Figure 12: Proof of Lemma 7.

2. The proof is similar to the one of Point 1.  $\square$

Therefore, the midpoint  $m$  of the third side of the triangle is either a vertex of  $P$  or on an edge of  $P$ . The goal of **Step 3** is to identify the sections of  $\Omega$  where the midpoint is a vertex and the sections of  $\Omega$  where the midpoint is on an edge of  $P$ .

**Step 3** Move the  $\omega$ -wedge with apex  $q$  clockwise along  $\Omega$  and maintain  $b, c$  and  $m$  as defined in **Step 2** (refer to Section 3). Collect all of the following three types of event points (refer to Figure 13).

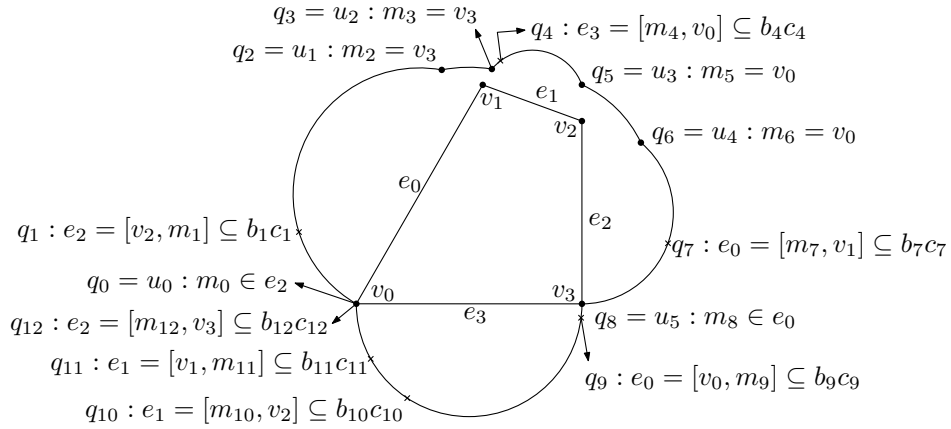


Figure 13: **Step 3:**  $q_0 = u_0 = v_0$  is such that  $m_0 \in e_2$ . Note that  $u_0 = v_0$  gives birth to two different event points, namely  $q_0$  and  $q_{12}$ . They correspond to the  $\frac{1}{2}\pi$ -wedge enclosing  $P$  and having  $u_0 \in \Gamma_0$  as a vertex, and the  $\frac{1}{2}\pi$ -wedge enclosing  $P$  and having  $u_0 \in \Gamma_5$  as a vertex.  $q_2 = u_1 : m_2 = v_3$  means that  $q_2 = u_1$ ,  $m_2 = v_3$  and  $b_2 c_2 \cap P = \{v_3\}$ . It is an event point of the first type. At such a place,  $m$  does not move even though  $q$  does.  $q_4 : e_3 = [m_4, v_0] \subseteq b_4 c_4$  means that  $m_4 = v_3$  and  $e_3 \subseteq b_4 c_4$ . It is an event point of the second type.  $q_9 : e_0 = [v_0, m_9] \subseteq b_9 c_9$  means that  $m_9 = v_1$  and  $e_0 \subseteq b_9 c_9$ . It is an event point of the third type.

**Type 1**  $q$  is on the intersection point of two consecutive circular arcs of  $\Omega$  for an  $i$  with  $0 \leq i \leq n' - 1$ . Formally,  $q = u_i$  for an  $i$  with  $0 \leq i \leq n' - 1$ .

**Type 2**  $q$  is such that the third side  $bc$  of the triangle is on an edge  $e_i$  of  $P$  (for an  $i$  with  $0 \leq i \leq n - 1$ ) and the midpoint  $m$  of  $bc$  is on the first vertex  $v_i$  of  $e_i$  (when the vertices of  $P$  are considered in clockwise order). Formally,  $q$  is such that  $m = v_i$  and  $e_i = [v_i, v_{i+1}] = [m, v_{i+1}] \subseteq bc$  for an  $i$  with  $0 \leq i \leq n - 1$ .

**Type 3**  $q$  is such that the third side  $bc$  of the triangle is on an edge  $e_i$  of  $P$  (for an  $i$  with  $0 \leq i \leq n - 1$ ) and the midpoint  $m$  of  $bc$  is on the last vertex  $v_{i+1}$  of  $e_i$  (when the vertices of  $P$  are considered in clockwise order). Formally,  $q$  is such that  $m = v_{i+1}$  and  $e_i = [v_i, v_{i+1}] = [v_i, m] \subseteq bc$  for an  $i$  with  $0 \leq i \leq n - 1$ .

For each event point  $q_i$ , save its type together with the location of  $m_i$ , the midpoint of the third side of the triangle.

It is easy to find the event points of the first type and the following two lemmas show how to find the event points of the second and third type. Lemma 8 helps identifying event points related to Case 1 of Lemma 7. As for Lemma 9, it helps identifying event points related to Case 2 of Lemma 7.

**Lemma 8.** *Let  $\Gamma = [v, w]$  be an arc of a circle,  $\Delta''$  be a line and  $v_j \in \Delta''$  be a point. It is possible to find a triangle  $\triangle qbc$  such that  $q \in \Gamma$ ,  $b, c \in \Delta''$  and  $v_j$  is the midpoint of  $bc$  (or decide that there is no such point) in  $O(1)$  time (refer to Figure 14).*

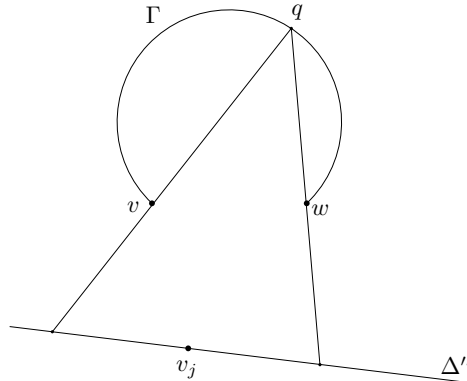


Figure 14: Illustration of Lemma 8.

**PROOF:** Without loss of generality,  $\Gamma$  is the locus of the point  $q$  such that  $\angle vqw = \omega$ . Hence we can take  $v = (0, 0)$ , and  $w = (2r \sin(\omega), 0)$ , where  $r$  is the radius of  $\Gamma$ . Let  $\alpha = \angle v_jvw$ ,  $\beta = \angle v_jvw$  and  $q$  be any point of  $\Gamma$  (refer to Figure 15). There are three cases to consider: (1) either  $\Delta''$  has strictly negative slope, (2)  $\Delta''$  has non-negative slope (3) or  $\Delta''$  is vertical.

- (1) Suppose  $\Delta''$  has strictly negative slope. The general equation for  $\Delta''$  is  $y = \lambda x + \mu$  ( $\lambda < 0$ ) and hence  $v_j = (s, \lambda s + \mu)$  for  $s \in \mathbb{R}$ . Note  $\theta = \angle qv_jw$ . Also let  $\Delta$  (respectively

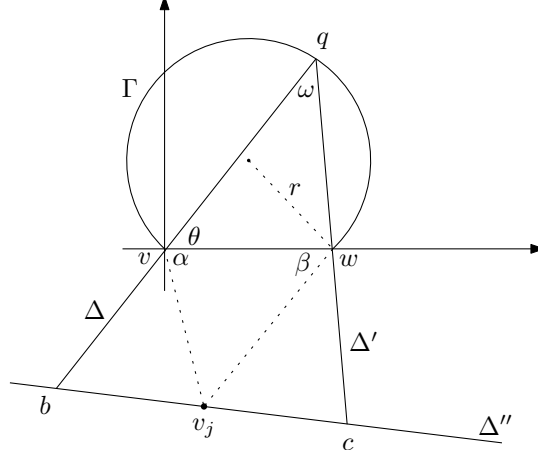


Figure 15: Proof of Lemma 8.

$\Delta'$ ) be the line through  $q$  and  $v$  (respectively through  $q$  and  $w$ ). Let  $b$  (respectively  $c$ ) be the intersection point of  $\Delta$  and  $\Delta''$  (respectively of  $\Delta'$  and  $\Delta''$ ).

Trivially  $0 \leq \theta \leq \pi - \omega$ , but there are other restrictions.  $c$  must be strictly to the right of  $v_j$ , so  $\beta - \omega < \theta$ . Also,  $b$  must be strictly to the left of  $v_j$ , so  $\theta < \pi - \alpha$ . Finally,  $q$  must not reach the point where the line through  $q$  and  $w$  is parallel to  $\Delta''$ . Hence  $\theta < \arctan(\lambda) - \omega$ . To summarize, we have

$$0 \leq \theta \leq \pi - \omega, \quad (1)$$

$$\beta - \omega < \theta < \min(\pi - \alpha, \arctan(\lambda) - \omega). \quad (2)$$

By elementary trigonometry, we get

$$q = (2r \cos(\theta) \sin(\theta + \omega), 2r \sin(\theta) \sin(\theta + \omega)).$$

Then, finding the general equation of the line through  $q$  and  $v$ , and of the line through  $q$  and  $w$ , we can calculate the general coordinates of  $b$  and  $c$ :

$$\begin{aligned} b &= \left( \frac{\mu \cot(\theta)}{1 - \lambda \cot(\theta)}, \frac{\mu}{1 - \lambda \cot(\theta)} \right), \\ c &= \left( \frac{2r \sin^2(\omega) \cot(\theta) + 2r \sin(\omega) \cos(\omega) + \mu \cos(\omega) \cot(\theta) - \mu \sin(\omega)}{\sin(\omega) \cot(\theta) + \cos(\omega) - \lambda \cos(\omega) \cot(\theta) + \lambda \sin(\omega)}, \right. \\ &\quad \left. \frac{(2r \sin(\omega) \lambda + \mu)(\sin(\omega) \cot(\theta) + \cos(\omega))}{\sin(\omega) \cot(\theta) + \cos(\omega) - \lambda \cos(\omega) \cot(\theta) + \lambda \sin(\omega)} \right). \end{aligned}$$

Therefore we want to find  $\theta$  such that  $v_j$  is the midpoint of  $bc$ , which leads to the equation

$$\frac{1}{2} \left( \frac{\mu \cot(\theta)}{1 - \lambda \cot(\theta)} + \frac{2r \sin^2(\omega) \cot(\theta) + 2r \sin(\omega) \cos(\omega) + \mu \cos(\omega) \cot(\theta) - \mu \sin(\omega)}{\sin(\omega) \cot(\theta) + \cos(\omega) - \lambda \cos(\omega) \cot(\theta) + \lambda \sin(\omega)} \right) = s$$

or

$$\begin{aligned}
& (2r \sin^2(\omega)\lambda + 2\mu \cos(\omega)\lambda - \mu \sin(\omega) - 2s\lambda \sin(\omega) + 2s\lambda^2 \cos(\omega))\cot^2(\theta) \\
& + (2r \sin(\omega) \cos(\omega)\lambda - 2r \sin^2(\omega) - 2\mu \cos(\omega) - 2\mu \sin(\omega)\lambda + 2s \sin(\omega) \\
& \quad - 4s \cos(\omega)\lambda - 2s\lambda^2 \sin(\omega))\cot(\theta) \\
& + (\mu \sin(\omega) - 2r \sin(\omega) \cos(\omega) + 2s \cos(\omega) + 2s\lambda \sin(\omega)) = 0 .
\end{aligned} \tag{3}$$

Since this is a quadratic equation in  $\cot(\theta)$ , it is solvable in constant time for  $\theta$  satisfying (1) and (2) (or it is possible to decide that there is no such solution in constant time).

- (2) Suppose  $\Delta''$  has non-negative slope. Using the notation of the previous case and a similar argument,  $\lambda \geq 0$  and

$$0 \leq \theta \leq \pi - \omega , \tag{4}$$

$$\max(\beta - \omega, \arctan(\lambda)) < \theta < \pi - \alpha . \tag{5}$$

We obtain the same quadratic equation (3) that is solvable in constant time for  $\theta$  satisfying (4) and (5) (or it is possible to decide that there is no such solution in constant time).

- (3) Suppose  $\Delta''$  is vertical. Therefore the general equation for  $\Delta''$  is  $x = s$  for  $s \in \mathbb{R}$ .  $0 < \omega < \frac{1}{2}\pi$  otherwise there is no solution. There are three subcases to consider: (3.1)  $s \leq 0$ , (3.2)  $s \in ]0, 2r \sin(\omega)[$  (3.3) or  $s \geq 2r \sin(\omega)$ .

- (3.1) Suppose  $\Delta''$  is vertical and  $s \leq 0$ . Let  $v_j = (s, t)$  for a  $t < 0$ . Trivially  $0 \leq \theta \leq \pi - \omega$ , but there are other restrictions. With the notation of the previous cases,  $w$  must be between  $q$  and  $c$ , therefore  $\theta < \frac{1}{2}\pi - \omega$ .  $b$  must be strictly above  $v_j$  so  $\theta < \pi - \alpha$ .  $c$  must be strictly under  $v_j$ , so  $\beta - \omega < \theta$ . It all sums up to

$$0 \leq \theta , \tag{6}$$

$$\beta - \omega < \theta < \min \left( \pi - \alpha, \frac{1}{2}\pi - \omega \right) . \tag{7}$$

By elementary trigonometry, we get

$$q = (2r \cos(\theta) \sin(\theta + \omega), 2r \sin(\theta) \sin(\theta + \omega)) .$$

Then, finding the general equation of the line through  $q$  and  $v$ , and of the line through  $q$  and  $w$ , we can calculate the general coordinates of  $b$  and  $c$ :

$$\begin{aligned}
b &= \left( s, \frac{s}{\cot \theta} \right) , \\
c &= \left( s, \frac{(s - 2r \sin(\omega))(\sin(\omega)\cot(\theta) + \cos(\omega))}{\cot(\theta) \cos(\omega) - \sin(\omega)} \right) .
\end{aligned}$$

Therefore we want to find  $\theta$  such that  $v_j$  is the midpoint of  $bc$ , which leads to the equation

$$\frac{1}{2} \left( \frac{s}{\cot(\theta)} + \frac{(s - 2r \sin(\omega))(\sin(\omega)\cot(\theta) + \cos(\omega))}{\cot(\theta) \cos(\omega) - \sin(\omega)} \right) = t$$

or

$$\begin{aligned} & (2r \sin^2(\omega) + t \cos(\omega) - s \sin(\omega)) \cot^2(\theta) \\ & + (2r \sin(\omega) \cos(\omega) - 2s \cos(\omega) - t \sin(\omega)) \cot(\theta) \\ & + s \sin(\omega) = 0 . \end{aligned} \tag{8}$$

Since this is a quadratic equation in  $\cot(\theta)$ , it is solvable in constant time for  $\theta$  satisfying (6) and (7) (or it is possible to decide that there is no such solution in constant time).

(3.2) Suppose  $\Delta''$  is vertical and  $s \in ]0, 2r \sin(\omega)[$ . In this case there is no solution.

(3.3) Suppose  $\Delta''$  is vertical and  $s \geq 2r \sin(\omega)$ . Using the notation of the case “ $\Delta''$  is vertical and  $s \leq 0$ ” and a similar argument,

$$\max \left( \beta - \omega, \frac{1}{2}\pi \right) < \theta < \pi - \alpha , \tag{9}$$

$$\theta \leq \pi - \omega . \tag{10}$$

We obtain the same quadratic equation (8) that is solvable in constant time for  $\theta$  satisfying (9) and (10) (or it is possible to decide that there is no such solution in constant time).  $\square$

**Lemma 9.** *Let  $q$  be a point,  $\Delta''$  be a line and  $v_j \in \Delta''$  be a point. It is possible to find the triangle  $\triangle qbc$  such that  $\angle bqc = \omega$ ,  $b, c \in \Delta''$  and  $v_j$  is the midpoint of  $bc$  in  $O(1)$  time (refer to Figure 16).*

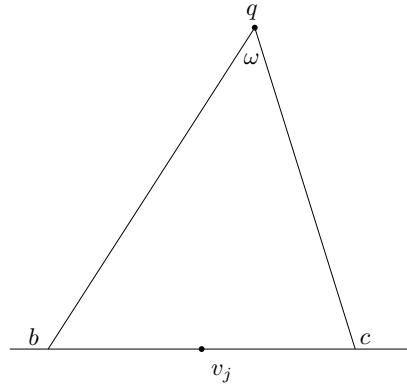


Figure 16: Illustration of Lemma 9.



PROOF: Without loss of generality,  $\Delta''$  is the  $x$ -axis,  $v_j = (0, 0)$ ,  $q = (s, t)$  with  $t > 0$  and  $b$  is to the left of  $c$ . Denote by  $d$  the center of the circumcircle of  $\triangle qbc$  and by  $r$ , its radius.

Suppose  $\omega \neq \frac{1}{2}\pi$ . By elementary geometry,  $d$  lies on the line segment bisector of  $bc$  and  $\angle bdv_j = \angle cdv_j = \omega$  (refer to Figure 17). Hence  $d = (0, h)$  for an  $h \in \mathbb{R}$ . Therefore, the

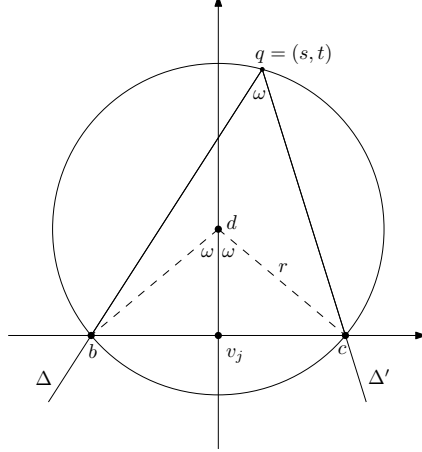


Figure 17: Proof of Lemma 9.

equation of the circumcircle is<sup>2</sup>

$$x^2 + (y - h)^2 = \frac{h^2}{\cos^2(\omega)} .$$

Since  $q$  is on this circle, then

$$s^2 + (t - h)^2 = \frac{h^2}{\cos^2(\omega)} ,$$

so

$$h = -\frac{\cos(\omega)}{\sin^2(\omega)} \left( t \cos(\omega) - \sqrt{s^2 \sin^2(\omega) + t^2} \right) ,$$

from which

$$\begin{aligned} b &= \left( t \cot(\omega) - \sqrt{s^2 \sin^2(\omega) + t^2} \csc(\omega), 0 \right) , \\ c &= \left( -t \cot(\omega) + \sqrt{s^2 \sin^2(\omega) + t^2} \csc(\omega), 0 \right) . \end{aligned}$$

If  $\omega = \frac{1}{2}\pi$ , a similar argument leads to the same solution. □

There are  $n' = O(n)$  event points of the first type. There are at most  $n$  event points of the second type because there are  $n$  edges. For the same reason, there are at most  $n$  event points of the third type. Therefore, **Step 3** (computing the event points) takes  $O(n)$  time.

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<sup>2</sup>We need the assumption  $\omega \neq \frac{1}{2}\pi$  here, otherwise  $\frac{h^2}{\cos^2(\omega)}$  is undefined.

## 5 Finding the Optimal Solution when the Apex is on a Circular Arc

Between any two consecutive event points, there is a single arc of a circle (that might be reduced to a single point if  $q_i$  is on one of the vertices of  $P$ ). Consider such a circular arc. As  $q$  moves along it, the minimum triangle enclosing  $P$  changes. In this section, we show how to compute the optimal triangle for a fixed arc of a circle between two event points. Four different cases can occur:

- either the circular arc is reduced to a single point or not. If the former is true, then the apex  $q$  is forced to stay on one of the vertices of  $P$ .
- Either the midpoint  $m$  is forced to stay on a vertex of  $P$  or not.

The following lemmas describe how to compute the minimum triangle enclosing  $P$  in these four different situations.

Lemma 10 describes how to compute the minimum enclosing triangle when  $q$  moves along an arc of a circle and  $m$  is forced to stay on one of the vertices of  $P$ .

**Lemma 10.** *Let  $\Gamma = [v, w]$  be an arc of a circle and  $v_j$  be a point. It is possible to find the triangle  $\triangle qbc$  of minimum area such that  $q \in \Gamma$ ,  $q, v$  and  $b$  lie on the same line,  $q, w$  and  $c$  lie on the same line, and  $v_j \in bc$  in  $O(1)$  time (refer to Figure 18).*

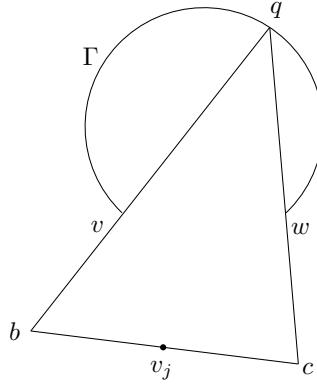


Figure 18: Illustration of Lemma 10.

**PROOF:** The strategy is to first fix a point  $q$  on  $\Gamma$  and then follow the proof of Lemma 5 in order to construct the minimum triangle for this fixed  $q$ . Then we move  $q$  along  $\Gamma$  while maintaining the minimum triangle. The smallest one among all these minimal triangles is optimal.

Without loss of generality,  $\Gamma$  is the locus of the point  $q$  such that  $\angle vqw = \omega$ . Hence we can take  $v = (0, 0)$  and  $w = (2r \sin(\omega), 0)$  where  $r$  is the radius of  $\Gamma$ . Let  $q$  be any point on  $\Gamma$  (refer to Figure 19). Note  $\gamma = |v_j v|$ ,  $\delta = |v_j w|$ ,  $\alpha = \angle v_j v w$  ( $0 < \alpha < \pi$ ),  $\beta = \angle v_j w v$  ( $0 < \beta < \pi$  and  $0 < \alpha + \beta < \pi$ ) and  $\theta = \angle q v w$ . Let  $w'$  be the orthogonal projection of  $w$



It all sums up to

$$\max \left( 0, \operatorname{arccot} \left( \frac{1}{2}(\cot(\beta) - \cot(\alpha)) \right) - \omega \right) \leq \theta , \quad (11)$$

$$\theta \leq \min \left( \pi - \omega, \operatorname{arccot} \left( \frac{1}{2}(\cot(\beta) - \cot(\alpha)) \right) \right) , \quad (12)$$

$$\beta - \omega < \theta < \pi - \alpha . \quad (13)$$

By the proof of Lemma 5, the area of  $\triangle qbc$  as a function of  $\theta$  is

$$\begin{aligned} \sigma(\theta) &= 2(s - t \cot(\omega))t \\ &= 2((\delta \cos(\theta - \beta) + 2r \cos(\omega) \sin(\theta)) - (\gamma \sin(\theta + \alpha)) \cot(\omega))(\gamma \sin(\theta + \alpha)) \\ &= 2\gamma \sin(\theta + \alpha)(\delta \cos(\theta - \beta) + 2r \cos(\omega) \sin(\theta) - \gamma \cot(\omega) \sin(\theta + \alpha)) . \end{aligned}$$

With the sine law applied on  $\triangle vv_jw$ , one gets

$$\begin{aligned} \gamma &= \frac{2r \sin(\omega) \sin(\beta)}{\sin(\alpha + \beta)} , \\ \delta &= \frac{2r \sin(\omega) \sin(\alpha)}{\sin(\alpha + \beta)} . \end{aligned}$$

Therefore,

$$\begin{aligned} \sigma(\theta) &= 2\gamma \sin(\theta + \alpha)(\delta \cos(\theta - \beta) + 2r \cos(\omega) \sin(\theta) - \gamma \cot(\omega) \sin(\theta + \alpha)) \\ &= -\frac{8r^2 \sin(\alpha) \sin(\beta) \sin(\omega)}{\sin^2(\alpha + \beta)} \sin(\beta - \omega - \theta) \sin(\alpha + \theta) . \end{aligned}$$

We have

$$\begin{aligned} \sigma'(\theta) &= -\frac{8r^2 \sin(\alpha) \sin(\beta) \sin(\omega)}{\sin^2(\alpha + \beta)} (\sin(\beta - \omega - \theta) \cos(\alpha + \theta) - \cos(\beta - \omega - \theta) \sin(\alpha + \theta)) \\ &= \frac{8r^2 \sin(\alpha) \sin(\beta) \sin(\omega)}{\sin^2(\alpha + \beta)} \sin(\alpha - \beta + \omega + 2\theta) . \end{aligned}$$

In order to find the candidates for minimum and maximum of  $\sigma$ , we need to solve

$$\sin(\alpha - \beta + \omega + 2\theta) = 0 \quad (14)$$

for  $\theta$  satisfying (11), (12) and (13).

We look for solutions to  $\alpha - \beta + \omega + 2\theta = k\pi$  for  $k \in \mathbb{Z}$ . At first sight, we have  $\theta = \frac{k\pi - \alpha + \beta - \omega}{2}$ , but we need to study this general solution in order for it to satisfy (11), (12) and (13). Putting this solution against these constraints, we get

$$\max \left( 0, \operatorname{arccot} \left( \frac{1}{2}(\cot(\beta) - \cot(\alpha)) \right) - \omega \right) \leq \frac{k\pi - \alpha + \beta - \omega}{2} , \quad (15)$$

$$\frac{k\pi - \alpha + \beta - \omega}{2} \leq \min \left( \pi - \omega, \operatorname{arccot} \left( \frac{1}{2}(\cot(\beta) - \cot(\alpha)) \right) \right) , \quad (16)$$

$$\beta - \omega < \frac{k\pi - \alpha + \beta - \omega}{2} < \pi - \alpha , \quad (17)$$

which simplify to

$$\max \left( \pm \left( -k\pi + \alpha - \beta + 2 \operatorname{arccot} \left( \frac{1}{2}(\cot(\beta) - \cot(\alpha)) \right) \right) \right) \leq \omega , \quad (18)$$

$$\omega \leq \min(k\pi - \alpha + \beta, -(k-2)\pi + \alpha - \beta) , \quad (19)$$

$$\max(-k\pi + \alpha + \beta, (k-2)\pi + \alpha + \beta) < \omega . \quad (20)$$

If  $k \leq 0$ , (20) and (19) lead to  $-k\pi + \alpha + \beta < \omega \leq k\pi - \alpha + \beta < -k\pi + \alpha + \beta$ , which is a contradiction. If  $k \geq 2$ , (20) and (19) lead to  $(k-2)\pi + \alpha + \beta < \omega \leq -(k-2)\pi + \alpha - \beta < (k-2)\pi + \alpha + \beta$ , which is a contradiction. If  $k = 1$ , then  $\theta = \frac{\pi - \alpha + \beta - \omega}{2}$  is a valid solution provided that

$$\max \left( \pm \left( \pi - \alpha + \beta - 2 \operatorname{arccot} \left( \frac{1}{2}(\cot(\beta) - \cot(\alpha)) \right) \right) \right) \leq \omega \leq \min(\pi - \alpha + \beta, \pi + \alpha - \beta) .$$

In this case,  $\theta = \frac{\pi - \alpha + \beta - \omega}{2}$  is a maximum as we show below.

$$\sigma''(\theta) = \frac{16r^2 \sin(\alpha) \sin(\beta) \sin(\omega)}{\sin^2(\alpha + \beta)} \cos(\alpha - \beta + \omega + 2\theta)$$

and

$$\begin{aligned} \sigma'' \left( \frac{\pi - \alpha + \beta - \omega}{2} \right) &= \frac{16r^2 \sin(\alpha) \sin(\beta) \sin(\omega) \cos(\pi)}{\sin^2(\alpha + \beta)} \\ &= -\frac{16r^2 \sin(\alpha) \sin(\beta) \sin(\omega)}{\sin^2(\alpha + \beta)} \\ &< 0 \end{aligned}$$

so  $\theta = \frac{\pi - \alpha + \beta - \omega}{2}$  is a maximum.

Otherwise, (14) has no solution, therefore  $\sigma(\theta)$  is monotonic. The conclusion is that the minimum of  $\sigma$  is at one (or both) of the extremities of the domain of  $\sigma$ .  $\square$

Looking at the proof of Lemma 10, it suggests that the minimum triangle might not exist. Indeed, if  $0 < \beta - \omega$  and  $\pi - \alpha < \pi - \omega$ , then  $\beta - \omega < \theta < \pi - \alpha$  by (11), (12) and (13). But since the minimum of  $\sigma$  is at one (or both) of the extremities of the domain of  $\sigma$ , then it does not exist. However, in the setting of the general problem, this does not occur. By Proposition 3, the minimum area triangle always exists for a given  $\omega$ -wedge so  $\theta$  will always vary inside an interval that includes its extremities.

Lemma 11 describes how to compute the minimum enclosing triangle when  $q$  moves along an arc of a circle and  $m$  is forced to stay on one of the edges of  $P$ .

**Lemma 11.** *Let  $\Gamma = [v, w]$  be an arc of a circle and  $\Delta''$  be a line. It is possible to find the point  $q \in \Gamma$  such that the line through  $qv$ , the line through  $qw$  and  $\Delta''$  form a triangle of minimum area in  $O(1)$  time (refer to Figure 20).*

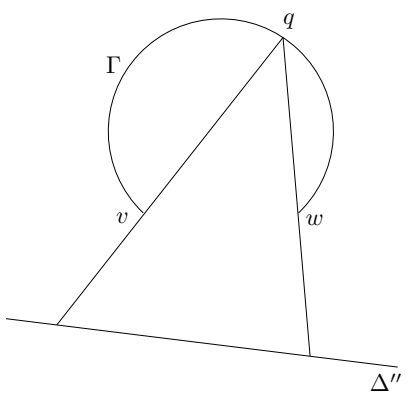


Figure 20: Illustration of Lemma 11.

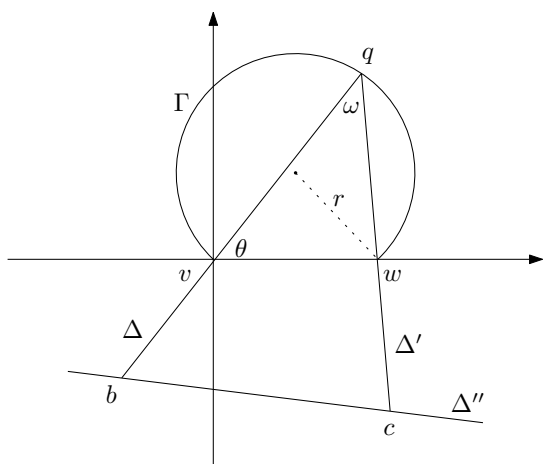


Figure 21: Proof of Lemma 11.

PROOF: Without loss of generality,  $\Gamma$  is the locus of the point  $q$  such that  $\angle vqw = \omega$ . Hence we can take  $v = (0, 0)$ , and  $w = (2r \sin(\omega), 0)$  where  $r$  is the radius of  $\Gamma$ . Let  $q$  be any point on  $\Gamma$  (refer to Figure 21). There are three cases to consider: (1)  $\Delta''$  has strictly negative slope, (2)  $\Delta''$  has non-negative slope or (3)  $\Delta''$  is vertical.

- (1) Suppose  $\Delta''$  has strictly negative slope. The general equation for  $\Delta''$  is  $y = \lambda x + \mu$  ( $\lambda < 0$ ). Note  $\theta = \angle qvw$ . Also note  $\Delta$  (respectively  $\Delta'$ ) the line through  $q$  and  $v$  (respectively through  $q$  and  $w$ ). Note  $b$  (respectively  $c$ ) the intersection point of  $\Delta$  and  $\Delta''$  (respectively of  $\Delta'$  and  $\Delta''$ ).

Trivially  $0 \leq \theta \leq \pi - \omega$ , but there is one more restriction.  $v$  must be between  $q$  and  $b$ , hence  $\theta < \arctan(\lambda) - \omega$ . It all sums up to

$$0 \leq \theta, \quad (21)$$

$$\theta \leq \pi - \omega, \quad (22)$$

$$\theta < \arctan(\lambda) - \omega. \quad (23)$$

By elementary trigonometry, we get

$$\begin{aligned} q &= (2r \cos(\theta) \sin(\theta + \omega), 2r \sin(\theta) \sin(\theta + \omega)) \\ &= \left( 2r \frac{\cos(\omega) \cot(\theta) + \sin(\omega) \cot^2(\theta)}{\cot^2(\theta) + 1}, 2r \frac{\cos(\omega) + \sin(\omega) \cot(\theta)}{\cot^2(\theta) + 1} \right). \end{aligned}$$

Then, finding the general equation of the line through  $q$  and  $v$ , and of the line through  $q$  and  $w$ , we can calculate the general coordinates of  $b$  and  $c$ :

$$\begin{aligned} b &= \left( \frac{\mu \cot(\theta)}{1 - \lambda \cot(\theta)}, \frac{\mu}{1 - \lambda \cot(\theta)} \right), \\ c &= \left( \frac{2r \sin^2(\omega) \cot(\theta) + 2r \sin(\omega) \cos(\omega) + \mu \cos(\omega) \cot(\theta) - \mu \sin(\omega)}{\sin(\omega) \cot(\theta) + \cos(\omega) - \lambda \cos(\omega) \cot(\theta) + \lambda \sin(\omega)}, \right. \\ &\quad \left. \frac{(2r \sin(\omega) \lambda + \mu)(\sin(\omega) \cot(\theta) + \cos(\omega))}{\sin(\omega) \cot(\theta) + \cos(\omega) - \lambda \cos(\omega) \cot(\theta) + \lambda \sin(\omega)} \right). \end{aligned}$$

Let  $X(\theta) = \cot(\theta)$ , therefore

$$\begin{aligned} q &= \left( 2r \frac{\sin(\omega) X^2 + \cos(\omega) X}{X^2 + 1}, 2r \frac{\sin(\omega) X + \cos(\omega)}{X^2 + 1} \right), \\ b &= \left( \frac{-\mu X}{\lambda X - 1}, \frac{-\mu}{\lambda X - 1} \right), \\ c &= \left( \frac{(2r \sin^2(\omega) + \mu \cos(\omega)) X + 2r \sin(\omega) \cos(\omega) - \mu \sin(\omega)}{(\sin(\omega) - \lambda \cos(\omega)) X + \cos(\omega) + \lambda \sin(\omega)}, \right. \\ &\quad \left. \frac{(2r \sin(\omega) \lambda + \mu)(\sin(\omega) X + \cos(\omega))}{(\sin(\omega) - \lambda \cos(\omega)) X + \cos(\omega) + \lambda \sin(\omega)} \right). \end{aligned}$$

Here is the area  $\sigma$  of  $\triangle qbc$ .

$$\begin{aligned}
& \sigma(X) \\
&= \frac{1}{2} |qb| |qc| \sin(\omega) \\
&= \frac{1}{2} \sin(\omega) \sqrt{\frac{((2\lambda r \sin(\omega) + \mu)X^2 + 2r(\lambda \cos(\omega) - \sin(\omega))X + \mu - 2r \cos(\omega))^2}{(\lambda X - 1)^2(1 + X^2)}} \times \\
&\quad \sqrt{\frac{((2\lambda r \sin(\omega) + \mu)X^2 + 2r(\lambda \cos(\omega) - \sin(\omega))X + \mu - 2r \cos(\omega))^2}{((\lambda \cos(\omega) - \sin(\omega))X - \lambda \sin(\omega) - \cos(\omega))^2(1 + X^2)}} \\
&= \frac{1}{2} \sin(\omega) \frac{((2\lambda r \sin(\omega) + \mu)X^2 + 2r(\lambda \cos(\omega) - \sin(\omega))X + \mu - 2r \cos(\omega))^2}{|(\lambda X - 1)((\lambda \cos(\omega) - \sin(\omega))X - \lambda \sin(\omega) - \cos(\omega))|(1 + X^2)} \\
&= \frac{1}{2} \sin(\omega) \frac{((2\lambda r \sin(\omega) + \mu)X^2 + 2r(\lambda \cos(\omega) - \sin(\omega))X + \mu - 2r \cos(\omega))^2}{(1 - \lambda X)(\lambda \sin(\omega) + \cos(\omega) - (\lambda \cos(\omega) - \sin(\omega))X)(1 + X^2)}
\end{aligned}$$

The reason why the absolute value  $|\cdot|$  disappeared is twofold. Firstly, by (23),  $\theta < \arctan(\lambda) - \omega < \arctan(\lambda) = \operatorname{arccot}(\frac{1}{\lambda})$ , so  $\lambda X = \lambda \cot(\theta) < 1$ . Secondly, by (23),  $\theta < \arctan(\lambda) - \omega$ , therefore

$$\begin{aligned}
\theta &< \arctan(\lambda) - \omega \\
\theta + \omega &< \arctan(\lambda) \\
\theta + \omega &< \operatorname{arccot}\left(\frac{1}{\lambda}\right) \\
\cot(\theta + \omega) &> \frac{1}{\lambda} && 0 < \theta + \omega < \pi \text{ by (21) and (22),} \\
\frac{\cot(\omega)\cot(\theta) - 1}{\cot(\theta) + \cot(\omega)} &> \frac{1}{\lambda} \\
\frac{\lambda \cot(\omega)\cot(\theta) - \lambda}{\cot(\theta) + \cot(\omega)} &< 1 && \lambda < 0 \text{ by the hypothesis,} \\
\lambda \cot(\omega)\cot(\theta) - \lambda &< \cot(\theta) + \cot(\omega) && 0 < \theta + \omega < \pi \text{ by (21) and (22),} \\
&&& \text{so } \cot(\theta) + \cot(\omega) > 0, \\
\lambda \cot(\omega)X - \lambda &< X + \cot(\omega) \\
(\lambda \cot(\omega) - 1)X &< \lambda + \cot(\omega) \\
(\lambda \cos(\omega) - \sin(\omega))X &< \lambda \sin(\omega) + \cos(\omega) && 0 < \omega < \pi \text{ by the hypothesis,} \\
&&& \text{so } \sin(\omega) > 0.
\end{aligned}$$

Now we need to find for what values of  $X$  is  $\sigma(X)$  minimum, wich means that we need to find for what  $X$  does  $\sigma'(X) = 0$ .

$$\sigma'(X)$$



$$\begin{aligned}
= & -\frac{1}{2}\sin(\omega) \left( (2\lambda r \sin(\omega) + \mu)X^2 + 2r(\lambda \cos(\omega) - \sin(\omega))X + \mu - 2r \cos(\omega) \right) \times \\
& \left( (-\mu \sin(\omega) + 2r \sin^2(\omega)\lambda + 2\lambda^3 r \sin^2(\omega) + 2\lambda\mu \cos(\omega) + \mu\lambda^2 \sin(\omega) \right. \\
& \quad \left. + 4\lambda^3 r \cos^2(\omega) - 4r\lambda^2 \cos(\omega) \sin(\omega))X^4 \right. \\
& \quad \left. + (-2r \sin^2(\omega) - 2\mu \cos(\omega) - 6\lambda^3 r \sin(\omega) \cos(\omega) + 2\lambda^2 r \sin^2(\omega) \right. \\
& \quad \left. + 2\mu\lambda^2 \cos(\omega) - 4\mu\lambda \sin(\omega) - 12\lambda^2 r \cos^2(\omega) + 10\lambda r \sin(\omega) \cos(\omega))X^3 \right. \\
& \quad \left. + 6r(\cos(\omega) + \lambda \sin(\omega))(-\sin(\omega) + 2\lambda \cos(\omega) + \lambda^2 \sin(\omega))X^2 \right. \\
& \quad \left. + (2r \sin^2(\omega) - 4r \cos^2(\omega) - 2\mu \cos(\omega) - 10\lambda^2 r \sin^2(\omega) - 4\mu\lambda \sin(\omega) \right. \\
& \quad \left. + 2\mu\lambda^2 \cos(\omega) + 2\lambda^3 r \sin(\omega) \cos(\omega) - 14\lambda r \sin(\omega) \cos(\omega))X \right. \\
& \quad \left. + (4r \sin^2(\omega)\lambda - \mu\lambda^2 \sin(\omega) - 2\lambda\mu \cos(\omega) + 2r \sin(\omega) \cos(\omega) \right. \\
& \quad \left. - 2r\lambda^2 \cos(\omega) \sin(\omega) + \mu \sin(\omega)) \right) \div \\
& ((1 - \lambda X)^2(-\lambda \sin(\omega) - \cos(\omega) + X\lambda \cos(\omega) - X \sin(\omega))^2(1 + X^2)^2)
\end{aligned}$$

Therefore  $\mu'(X) = 0$  if and only if

$$(2\lambda r \sin(\omega) + \mu)X^2 + 2r(\lambda \cos(\omega) - \sin(\omega))X + \mu - 2r \cos(\omega) = 0 \quad (24)$$

or

$$\begin{aligned}
& (-\mu \sin(\omega) + 2r \sin^2(\omega)\lambda + 2\lambda^3 r \sin^2(\omega) + 2\lambda\mu \cos(\omega) + \mu\lambda^2 \sin(\omega) \\
& \quad + 4\lambda^3 r \cos^2(\omega) - 4r\lambda^2 \cos(\omega) \sin(\omega))X^4 \\
& \quad + (-2r \sin^2(\omega) - 2\mu \cos(\omega) - 6\lambda^3 r \sin(\omega) \cos(\omega) + 2\lambda^2 r \sin^2(\omega) \\
& \quad + 2\mu\lambda^2 \cos(\omega) - 4\mu\lambda \sin(\omega) - 12\lambda^2 r \cos^2(\omega) + 10\lambda r \sin(\omega) \cos(\omega))X^3 \\
& \quad + 6r(\cos(\omega) + \lambda \sin(\omega))(-\sin(\omega) + 2\lambda \cos(\omega) + \lambda^2 \sin(\omega))X^2 \\
& \quad + (2r \sin^2(\omega) - 4r \cos^2(\omega) - 2\mu \cos(\omega) - 10\lambda^2 r \sin^2(\omega) - 4\mu\lambda \sin(\omega) \\
& \quad + 2\mu\lambda^2 \cos(\omega) + 2\lambda^3 r \sin(\omega) \cos(\omega) - 14\lambda r \sin(\omega) \cos(\omega))X \\
& \quad + (4r \sin^2(\omega)\lambda - \mu\lambda^2 \sin(\omega) - 2\lambda\mu \cos(\omega) + 2r \sin(\omega) \cos(\omega) \\
& \quad - 2r\lambda^2 \cos(\omega) \sin(\omega) + \mu \sin(\omega)) = 0 . \quad (25)
\end{aligned}$$

The first equation is quadratic in  $X = \cot(\theta)$ , therefore it can be solved in constant time for  $X$  and then it remains to solve for  $\theta$ . The second equation is quartic in  $X = \cot(\theta)$ , therefore it can be solved<sup>3</sup> in constant time for  $X$  and then it remains to solve for  $\theta$ .

Actually, what we are really looking for is  $\mu'(X(\theta)) = 0$ . Since

$$\mu'(X(\theta)) = \mu'(X)X'(\theta) = \mu'(X)\csc(\theta)\cot(\theta) ,$$

$\theta = \frac{1}{2}\pi$  is also a candidate. And of course, the extremities of the domain of  $\theta$  are also candidates.

---

<sup>3</sup>By Galois theory, polynomials of degree  $d \leq 4$  can be solved exactly in constant time. Refer to [DF03]

Overall, we get at most nine candidates for minimum (at most two from the quadratic equation, at most four from the quartic equation,  $\theta = \frac{1}{2}\pi$  and the extremities.) So it can be solved exactly in constant time by taking the smallest of the nine.

- (2) Suppose  $\Delta''$  has non-negative slope. Using the notation of the previous case and a similar argument,  $\lambda \geq 0$  and

$$0 \leq \theta, \quad (26)$$

$$\arctan(\lambda) < \theta, \quad (27)$$

$$\theta \leq \pi - \omega. \quad (28)$$

We obtain the same equations (24) and (25), and therefore the candidates are the same nine ones. So it can be solved exactly in constant time by taking the candidates that minimize  $\mu(\theta)$ .

- (3) Suppose  $\Delta''$  is vertical. The general equation of  $\Delta$  is  $x = s$  for  $s \in \mathbb{R}$ .  $0 < \omega < \frac{1}{2}\pi$  otherwise there is no solution. Using the notation of the previous cases, there are three subcases to consider: (3.1)  $s \leq 0$ , (3.2)  $s \in ]0, 2r \sin(\omega)[$  or (3.3)  $s \geq 2r \sin(\omega)$ .

- (3.1) Suppose  $\Delta''$  is vertical and  $s \leq 0$ . Trivially  $0 \leq \theta \leq \pi - \omega$ , but there is one more restriction.  $v$  must be between  $q$  and  $b$ , hence  $\theta < \frac{1}{2}\pi - \omega$ . It all sums up to

$$0 \leq \theta, \quad (29)$$

$$\theta < \frac{1}{2}\pi - \omega. \quad (30)$$

By elementary trigonometry, we get

$$\begin{aligned} q &= (2r \cos(\theta) \sin(\theta + \omega), 2r \sin(\theta) \sin(\theta + \omega)) \\ &= \left( 2r \frac{\cos(\omega) \cot(\theta) + \sin(\omega) \cot^2(\theta)}{\cot^2(\theta) + 1}, 2r \frac{\cos(\omega) + \sin(\omega) \cot(\theta)}{\cot^2(\theta) + 1} \right). \end{aligned}$$

Then, finding the general equation of the line through  $q$  and  $v$ , and of the line through  $q$  and  $w$ , we can calculate the general coordinates of  $b$  and  $c$ :

$$\begin{aligned} b &= \left( s, \frac{s}{\cot(\theta)} \right), \\ c &= \left( s, \frac{(s - 2r \sin(\omega))(\sin(\omega) \cot(\theta) + \cos(\omega))}{\cos(\omega) \cot(\theta) - \sin(\omega)} \right). \end{aligned}$$

We note  $X(\theta) = \cot(\theta)$ , therefore

$$\begin{aligned} q &= \left( 2r \frac{\sin(\omega) X^2 + \cos(\omega) X}{X^2 + 1}, 2r \frac{\sin(\omega) X + \cos(\omega)}{X^2 + 1} \right), \\ b &= \left( s, \frac{s}{X} \right), \\ c &= \left( s, \frac{(s - 2r \sin(\omega))(\sin(\omega) X + \cos(\omega))}{\cos(\omega) X - \sin(\omega)} \right). \end{aligned}$$

Here is the area  $\sigma$  of  $\triangle qbc$ .

$$\begin{aligned}
& \sigma(X) \\
&= \frac{1}{2} |qb| |qc| \sin(\omega) \\
&= \frac{1}{2} \sin(\omega) \sqrt{\frac{((2r \sin(\omega) - s)X^2 + 2r \cos(\omega)X - a)^2}{X^2(X^2 + 1)}} \times \\
&\quad \sqrt{\frac{((2r \sin(\omega) - s)X^2 + 2r \cos(\omega)X - s)^2}{(\cos(\omega)X - \sin(\omega))^2(X^2 + 1)}} \\
&= \frac{1}{2} \sin(\omega) \frac{((2r \sin(\omega) - s)X^2 + 2r \cos(\omega)X - s)^2}{|X(\cos(\omega)X - \sin(\omega))| (X^2 + 1)} \\
&= \frac{1}{2} \sin(\omega) \frac{((2r \sin(\omega) - s)X^2 + 2r \cos(\omega)X - s)^2}{X(\cos(\omega)X - \sin(\omega))(X^2 + 1)}
\end{aligned}$$

The reason why the absolute value  $|\cdot|$  disappeared is twofold. Firstly, by the hypothesis,  $0 < \theta < \frac{1}{2}\pi$ , so  $X = \cot(\theta) > 0$ . Secondly, by (30),  $\theta < \frac{1}{2}\pi - \omega$ , therefore

$$\begin{aligned}
\theta &< \frac{1}{2}\pi - \omega \\
\cot(\theta) &> \cot\left(\frac{1}{2}\pi - \omega\right) \\
\cot(\theta) &> \tan(\omega) \\
\cos(\omega)\cot(\theta) &> \sin(\omega) \\
\cos(\omega)\cot(\theta) - \sin(\omega) &> 0 \\
\cos(\omega)X - \sin(\omega) &> 0 .
\end{aligned}$$

Now we need to find for what values of  $X$  where  $\sigma(X)$  is minimum, wich means that we need to find for wich  $X$  does  $\sigma'(X) = 0$ .

$$\begin{aligned}
& \sigma'(X) \\
&= -\frac{1}{2} \sin(\omega) ((2r \sin(\omega) - s)X^2 + 2r \cos(\omega)X - s) \times \\
&\quad ((-s \sin(\omega) + 2r \sin^2(\omega) + 4r \cos^2(\omega))X^4 - 2 \cos(\omega)(s + 3r \sin(\omega))X^3 \\
&\quad + 6r \sin^2(\omega)X^2 + 2 \cos(\omega)(r \sin(\omega) - s)X + s \sin(\omega)) \div \\
&\quad (X^2(\cos(\omega)X - \sin(\omega))^2(X^2 + 1)^2)
\end{aligned}$$

Therefore  $\sigma'(X) = 0$  if and only if

$$(2r \sin(\omega) - s)X^2 + 2r \cos(\omega)X - s = 0 \quad (31)$$

or

$$(-s \sin(\omega) + 2r \sin^2(\omega) + 4r \cos^2(\omega))X^4 - 2 \cos(\omega)(s + 3r \sin(\omega))X^3 + 6r \sin^2(\omega)X^2 + 2 \cos(\omega)(r \sin(\omega) - s)X + s \sin(\omega) = 0 . \quad (32)$$

The first equation is quadratic in  $X = \cot(\theta)$ , therefore it can be solved in constant time for  $X$  and then it remains to solve for  $\theta$ . The second equation is quartic in  $X = \cot(\theta)$ , therefore it can be solved in constant time for  $X$  and then it remains to solve for  $\theta$ .

Actually, what we are really looking for is  $\mu'(X(\theta)) = 0$ . Since  $\mu'(X(\theta)) = \mu'(X)X'(\theta) = \mu'(X)\csc(\theta)\cot(\theta)$ ,  $\theta = \frac{1}{2}\pi$  is also a candidate. And of course, the extremities of the domain of  $\theta$  are also candidates.

Overall, we get at most nine candidates for minimum (at most two from the quadratic equation, at most four from the quartic equation,  $\theta = \frac{1}{2}\pi$  and the extremities.) So it can be solved exactly in constant time by taking the candidates that minimize  $\mu(\theta)$ .

(3.2) Suppose  $\Delta''$  is vertical and  $s \in ]0, 2r \sin(\omega)[$ . In this case there is no solution.

(3.3) Suppose  $\Delta''$  is vertical and  $s \geq 2r \sin(\omega)$ . Using the notation of the case “ $\Delta''$  is vertical and  $s \leq 0$ ” and a similar argument,

$$\frac{1}{2}\pi < \theta , \quad (33)$$

$$\theta \leq \pi - \omega . \quad (34)$$

We obtain the same equations (31) and (32), and therefore the candidates are the same nine ones. So it can be solved exactly in constant time by taking the candidates that minimize  $\mu(\theta)$ .  $\square$

Lemma 12 describes how to compute the minimum enclosing triangle when  $q$  is forced to stay on one of the vertices of  $P$  and  $m$  is forced to stay on another one of the vertices of  $P$ .

**Lemma 12.** *Let  $q$  and  $v_j$  be points. It is possible to find the triangle  $\triangle qbc$  of minimum area such that  $v_j$  is the midpoint of  $bc$  and  $\angle bqc = \omega$  in  $O(1)$  time, provided that  $qb$  and  $qc$  are restrained to a closed wedge  $W'$  strictly included in a  $2\omega$ -wedge  $W$  such that the angle bisector of  $W$  contains  $v_j$  (refer to Figure 22).*

PROOF: Without loss of generality,  $q = (0, 0)$  and  $v_j = (\mu \cot(\omega), \mu)$  with  $\mu > 0$ . Let  $\Delta$  (respectively  $\Delta'$ ) be the line of general equation  $x - \cot(\theta)y = 0$  (respectively  $x - \cot(\theta + \omega)y = 0$ ) for  $0 < \theta < \omega$ . (refer to Figure 23). The  $\omega$ -wedges created by  $\Delta$  and  $\Delta'$  for  $0 < \theta < \omega$  are all possible  $\omega$ -wedges of vertex  $q$  that enclose  $v_j$ . Therefore, we have to find the value of  $\theta$  such that the triangle  $\triangle qbc$  (where  $b \in \Delta$ ,  $c \in \Delta'$  and  $v_j$  is the midpoint of  $bc$ ) is of minimum area.

Let  $\Delta''$  be the line through  $q$  that is perpendicular to  $\Delta$ . The general equation of  $\Delta''$  is  $\cot(\theta)x + y = 0$ .

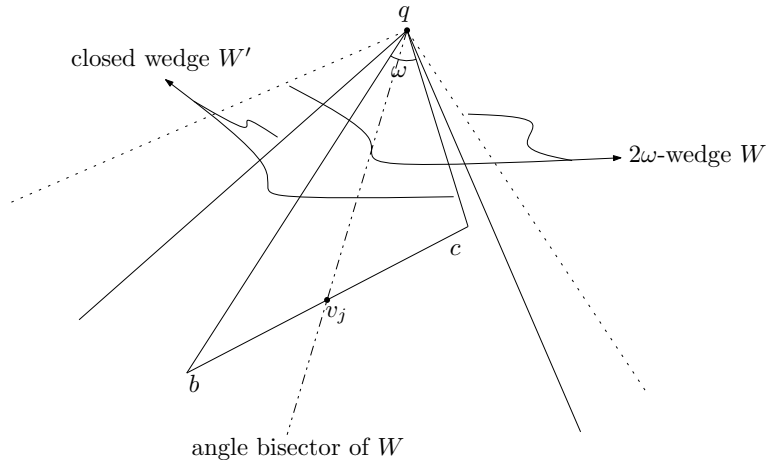


Figure 22: Illustration of Lemma 12.

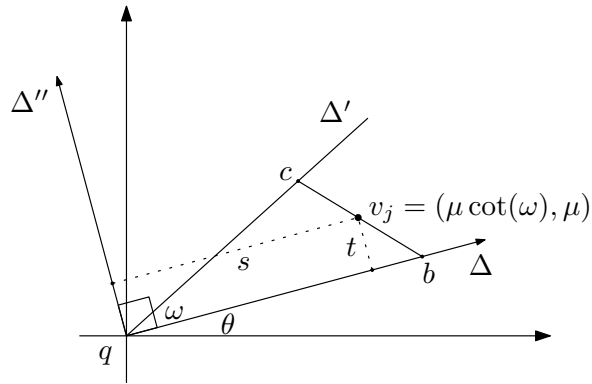


Figure 23: Proof of Lemma 12.

Fix  $\theta$  and apply Lemma 5 where  $v := v_j$  and the coordinate system is the one created by  $\Delta$  and  $\Delta''$ . Let  $s$  (respectively  $t$ ) be the coordinate of  $v_j$  with respect to the  $\Delta$  axis (respectively with respect to the  $\Delta''$  axis). Then, following the proof of Lemma 5, let  $b$  (respectively  $c$ ) be the point of  $\Delta$  (respectively of  $\Delta'$ ) such that  $|qb| = 2(s - t\cot(\omega))$  (respectively  $|qc| = 2t\csc(\omega)$ ).

Here is the area  $\sigma$  of  $\triangle qbc$ .

$$\begin{aligned}
\sigma(\theta) &= \frac{1}{2}|qb||qc|\sin(\omega) \\
&= \frac{1}{2}2(s - t\cot(\omega))2t\csc(\omega)\sin(\omega) \\
&= 2t(s - t\cot(\omega)) \\
&= 2(\cos(\theta) - \sin(\theta)\cot(\omega))\mu \times \\
&\quad ((\sin(\theta) + \cos(\theta)\cot(\omega))\mu - (\cos(\theta) - \sin(\theta)\cot(\omega))\mu\cot(\omega)) \\
&= 2\mu^2\csc^3(\omega)\sin(\omega - \theta)\sin(\theta)
\end{aligned}$$

Now we need to find for what values of  $\theta$  is  $\sigma(\theta)$  minimum, which means that we need to find for what  $\theta$  does  $\sigma'(\theta) = 0$ .

$$\sigma'(\theta) = 2\mu^2\csc^3(\omega)\sin(\omega - 2\theta)$$

so  $\sigma'(\theta) = 0$  if and only if

$$\sin(\omega - 2\theta) = 0,$$

therefore<sup>4</sup>  $\theta = \frac{1}{2}\omega$ .

Is  $\theta = \frac{1}{2}\omega$  a minimum or a maximum?

$$\sigma''(\theta) = -4\mu^2\csc^3(\omega)\cos(\omega - 2\theta)$$

so

$$\begin{aligned}
&\sigma''\left(\frac{1}{2}\omega\right) \\
&= -4\mu^2\csc^3(\omega) \\
&< 0
\end{aligned}$$

because  $0 < \omega < \pi$ . Hence  $\theta = \frac{1}{2}\omega$  is the unique (local and global) maximum.

Therefore, if the domain of  $\theta$  is restricted to  $0 < \phi_1 \leq \theta \leq \phi_2 < \pi$ , then the minimum of  $\mu(\theta)$  is either at  $\theta = \phi_1$  or at  $\theta = \phi_2$  and it can be computed in constant time.  $\square$

Lemma 13 describes how to compute the minimum enclosing triangle when  $q$  is forced to stay on one of the vertices of  $P$  and  $m$  is forced to stay on one of the edges of  $P$ .

**Lemma 13.** *Let  $q$  be a point and  $\Delta''$  be a line. The triangle  $\triangle qbc$  of minimum area such that  $b, c \in \Delta''$  and  $\angle bq c = \omega$  is isosceles and can be found in  $O(1)$  time (refer to Figure 24).*

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<sup>4</sup>It corresponds to the case where  $\triangle qbc$  is isosceles.

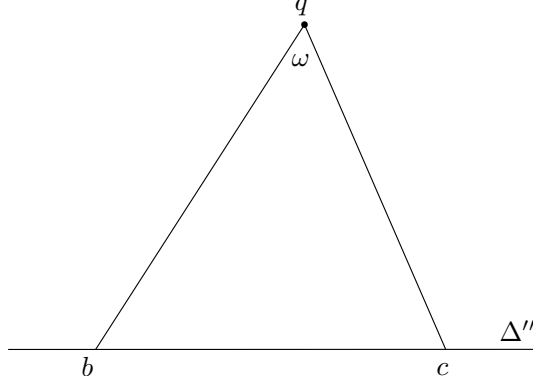


Figure 24: Illustration of Lemma 13.

PROOF: Without loss of generality,  $\Delta''$  is horizontal,  $q$  is above  $\Delta''$  and  $b$  is to the left of  $c$ . We need two intermediate results which follow from elementary geometry.

### Intermediate Results

1. Let  $\triangle qbc$  and  $\triangle q'b'c'$  be two isosceles triangles such that  $|qb| = |qc|$ ,  $|q'b'| = |q'c'|$  and  $\angle bqc = \angle b'q'c' = \omega$  (refer to Figure 25). Denote by  $h_q$  (respectively by  $h_{q'}$ ) the height

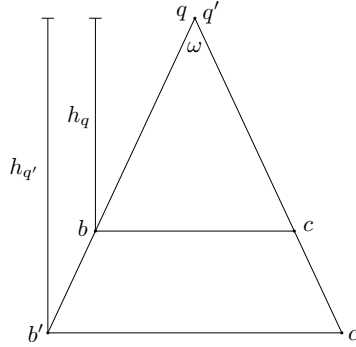


Figure 25: Intermediate result 1.

of  $\triangle qbc$  (respectively of  $\triangle q'b'c'$ ) relative to  $bc$  (respectively to  $b'c'$ ). If  $h_{q'} > h_q$ , then  $|b'c'| > |bc|$ .

2. Let  $\triangle qbc$  be an isosceles triangle such that  $|qb| = |qc|$  and  $\angle bqc = \omega$ . Let  $\triangle q'bc$  be a triangle such that  $\angle bq'c = \omega$  (refer to Figure 26). Denote by  $h_q$  (respectively by  $h_{q'}$ ) the height of  $\triangle qbc$  (respectively of  $\triangle q'bc$ ) relative to  $bc$  (respectively to  $bc$ ). Then  $h_{q'} \leq h_q$ .

We are now ready for the main demonstration. Let  $\triangle qb'c'$  be any triangle such that  $b', c' \in \Delta''$ ,  $\angle b'qc' = \omega$  and  $b'$  is to the left of  $c'$  (refer to Figure 27). Let  $\triangle q'b'c'$  be the isosceles triangle such that  $|q'b'| = |q'c'|$  and  $\angle b'q'c' = \omega$ . Denote by  $h_q$  (respectively by  $h_{q'}$ ) the height of  $\triangle qbc$  (respectively of  $\triangle q'b'c'$ ) relative to  $bc$  (respectively to  $b'c'$ ). By Intermediate result 2,  $h_{q'} \geq h_q$ , so  $|b'c'| \geq |bc|$  by Intermediate result 1. Since  $\triangle qb'c'$  and  $\triangle qbc$  have the same height, then  $\triangle qbc$  is of minimum area.  $\square$

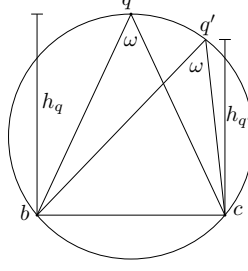


Figure 26: Intermediate result 2.

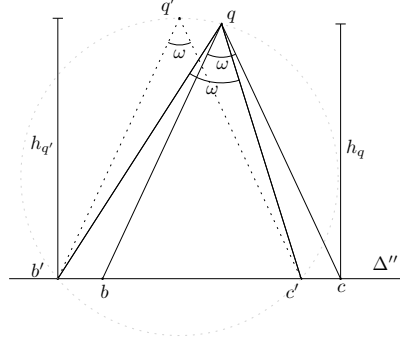


Figure 27: Proof of Lemma 13.

**Step 4** For each pair of consecutive event points, compute the minimum triangle enclosing  $P$  and having a vertex on the corresponding arc of a circle.

Since there are  $O(n)$  event points, **Step 4** takes  $O(n)$ .

## 6 Putting it All Together

The vertex that subtends the angle  $\omega$  of the optimal triangle enclosing  $P$  has to be on the  $\omega$ -cloud  $\Omega$ .  $\Omega$  was computed in **Step 1** in  $O(n)$  time (refer to Section 2). In **Step 2** (refer to Section 3), we fixed an  $\omega$ -wedge  $W$  and computed the minimum triangle that can be constructed with  $W$  in  $O(n)$  time. This triangle is such that the midpoint  $m$  of its third side touches  $P$  (refer to Proposition 3 and Corollary 6). Then we divided  $\Omega$  into a linear number of pieces in **Step 3** (refer to Section 4). Within each of these pieces, an optimal triangle can be computed in  $O(1)$  time. This was done in **Step 4** (refer to Section 5).

**Step 5** Find the smallest triangles among those calculated in **Step 4**.

Since there are  $O(n)$  event points, **Step 5** takes  $O(n)$  time. Since each step takes no longer than  $O(n)$  time, then the algorithm takes  $O(n)$  time. If the input is a set of points, compute the convex hull and then apply this algorithm. In this situation, the computation clearly takes  $O(n \log(n))$  time.

We summarize the final algorithm.



**Algorithm 14.** (*Minimum Enclosing Triangle with a Fixed Angle*)

- *INPUT:* A finite set  $S$  of points in the plane and an angle  $0 < \omega < \pi$ .
  - *OUTPUT:* All triangles with minimum area having an angle of  $\omega$  that enclose  $S$ .
0. Compute the convex hull of  $S$  and denote it by  $P$ . We denote the edges and the vertices of  $P$  in clockwise order by  $e_i$  and  $v_i$  for  $0 \leq i \leq n-1$  (all index manipulation is modulo  $n$ ).
  1. Compute the  $\omega$ -cloud around  $P$  and denote it by  $\Omega$  (refer to Section 2).  $\Omega$  consists of  $n' = O(n)$  circular arcs that we denote in clockwise order by  $\Gamma_i$  for  $0 \leq i \leq n' - 1$ . The intersection point of  $\Gamma_i$  and  $\Gamma_{i+1}$  is denoted by  $u_{i+1}$  for  $0 \leq i \leq n' - 1$ .
  2. Let  $q \in \Gamma_0$  be such that  $q = u_0$ . Consider the  $\omega$ -wedge  $W = \mathcal{W}(\omega, q, \Delta, \Delta')$  that encloses  $P$ . Apply Algorithm 4 with  $P$  and  $W$  (refer to Section 3). Let  $\Delta qbc$  be the output of Algorithm 4 and denote by  $m$  the midpoint of segment  $bc$ .
  3. Move  $q$  clockwise along  $\Omega$  and maintain  $W$ ,  $b$ ,  $c$  and  $m$  as defined in 2. Collect all of the following three types of event points (see Section 4 for formal definition)

**Type 1**  $q$  is on the intersection point of two consecutive circular arcs of  $\Omega$  for an  $i$  with  $0 \leq i \leq n' - 1$ .

**Type 2**  $q$  is such that the third side  $bc$  of the triangle is on an edge  $e_i$  of  $P$  (for an  $i$  with  $0 \leq i \leq n - 1$ ) and the midpoint  $m$  of  $bc$  is on the first vertex  $v_i$  of  $e_i$  (when the vertices of  $P$  are considered in clockwise order).

**Type 3**  $q$  is such that the third side  $bc$  of the triangle is on an edge  $e_i$  of  $P$  (for an  $i$  with  $0 \leq i \leq n - 1$ ) and the midpoint  $m$  of  $bc$  is on the last vertex  $v_{i+1}$  of  $e_i$  (when the vertices of  $P$  are considered in clockwise order).

4. For each pair of consecutive event points computed in 3, which we denote by  $(q', q'')$ , there is a single arc of a circle. This circular arc might be reduced to a single point if two consecutive event points are on one of the vertices of  $P$ . When  $q$  moves on such an arc, either  $m$  is forced to stay on one of the vertices of  $P$  or  $m$  is forced to stay on one of the edges of  $P$  (see Sections 2 and 4 for complete discussion). For each pair  $(q', q'')$  of consecutive event points,
  - if the circular arc between  $q'$  and  $q''$  is not reduced to a single point and if  $m$  is forced to stay on one of the vertices of  $P$ , store the triangle defined by Lemma 10 (refer to Section 5).
  - If the circular arc between  $q'$  and  $q''$  is not reduced to a single point and if  $m$  is forced to stay on one of the edges of  $P$ , store the triangle defined by Lemma 11 (refer to Section 5).

- If the circular arc between  $q'$  and  $q''$  is reduced to a single point and if  $m$  is forced to stay on one of the vertices of  $P$ , store the triangle defined by Lemma 12 (refer to Section 5).
- If the circular arc between  $q'$  and  $q''$  is reduced to a single point and if  $m$  is forced to stay on one of the edges of  $P$ , store the triangle defined by Lemma 13 (refer to Section 5).

5. Return the smallest triangles among those stored in 4.

## 7 A Note on the Complexity of the Solution

Notice that one of the cases in Section 5 required us to find the roots of a fourth degree polynomial (refer to Lemma 11). One may ask whether or not this is necessary or if one can avoid finding the roots of such a polynomial to solve this problem. In this section, we show that it is unavoidable in certain situations, by providing an example of a set of points where the optimal solution lies on the root of an irreducible fourth degree polynomial.

Consider the quadrilateral  $[abcd]$  where  $a = (0,0)$ ,  $b = (2,0)$ ,  $c = (2, -\frac{3}{2})$  and  $d = (-4\frac{4\sqrt{3}-1}{47}, 4\frac{\sqrt{3}-12}{47})$ , and take  $\omega = \frac{1}{2}\pi$  (hence we are looking for the minimum enclosing right triangle). It turns out that the optimal right triangle is such that the right angle is on  $\Gamma$  (refer to Figure 28) and the hypotenuse is on  $cd$ . Therefore, we need to solve (25) with

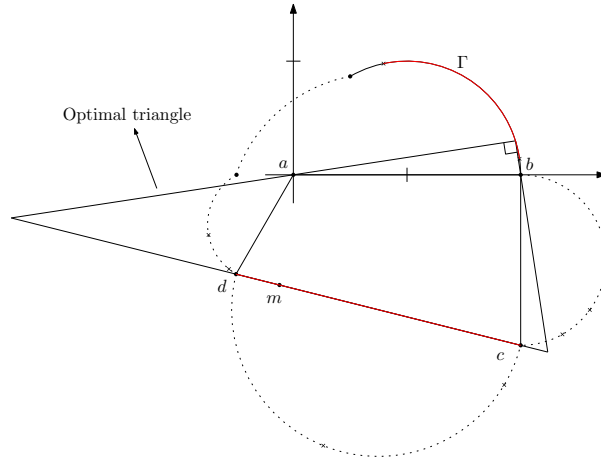


Figure 28: Example where the optimal solution involves the roots of a quartic equation.

$\mu := -1$ ,  $\lambda := -\frac{1}{4}$  and  $r := 1$ . It leads to the following quartic equation.

$$p_4(X) = 13X^4 - 92X^3 + 45X^2 + 12X - 62 = 0$$

This equation admits two real solutions  $X_1$  and  $X_2$ . Note

$$\alpha = \sqrt[3]{-1107 + 6\sqrt{34134}} ,$$

$$\beta = \sqrt{1726 - \frac{1105}{\alpha} + \frac{221}{3}\alpha} ,$$

then

$$\begin{aligned} X_1 &= \frac{23}{13} + \frac{1}{26}\beta - \frac{1}{2}\sqrt{\frac{3452}{169} - \frac{17}{39}\alpha + \frac{85}{13\alpha} + \frac{136796}{169\beta}} \approx -0.761694 , \\ X_2 &= \frac{23}{13} + \frac{1}{26}\beta + \frac{1}{2}\sqrt{\frac{3452}{169} - \frac{17}{39}\alpha + \frac{85}{13\alpha} + \frac{136796}{169\beta}} \approx 6.543373 . \end{aligned}$$

Solving  $\cot(\theta) = X_1$  and  $\cot(\theta) = X_2$ , then testing against  $\sigma(\theta)$  (refer to the proof of Lemma 11), one finds that  $\theta \approx \operatorname{arccot}(6.543373) \approx 0.048273\pi$  minimizes  $\sigma(\theta)$ .

Since  $p_4(X)$  and its resolvent cubic

$$\rho(X) = X^3 - \frac{45}{13}X^2 + \frac{2120}{169}X + \frac{377816}{2197}$$

are irreducible over  $\mathbb{Q}[X]$ , the algebraic expressions for the roots of  $p_4$  must be written with square roots and cubic roots. Moreover, they cannot be simplified (see [DF03]).

Therefore, in general, we cannot avoid square roots nor cubic roots in the computation of the minimum enclosing area triangle with a fixed angle.

## 8 Conclusion

We have shown how to compute all triangles of minimum area with a fixed angle  $0 < \omega < \pi$  that enclose a point set. It would be interesting to see if some of these techniques generalize to other settings or other optimization criteria. For example, finding the smallest area tetrahedron with a fixed solid angle of a set of points in three dimensions.

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